3401

**2012**



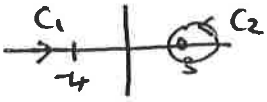
1 a) $xy'' + (2+x)y' - (6+12x)y = 0$ , $y = \int_c e^{xt} f(t)\,dt$ requires

$$\int_c x\begin{pmatrix}t^2+t\\-12\end{pmatrix} f\,e^{xt} + (2t-6) f e^{xt}\,dt = 0 \Rightarrow \Big[(t-3)(t+4)e^{xt} f\Big]_c$$

$$+ \int_c \big[(2t-6) - (2t+1)\big] f e^{xt} - (t-3)(t+4) f' e^{xt}\,dt = 0$$

So, set $\dfrac{f'}{f} = \dfrac{-7}{(t-3)(t+4)} = \dfrac{-1}{(t-3)} + \dfrac{1}{(t+4)} \Rightarrow f = \dfrac{(t+4)}{(t-3)}$ & $\Big[(t+4)^2 e^{xt}\Big]_c = 0$

& $y = \int_c e^{xt}\dfrac{(t+4)}{(t-3)}\,dt$




& $y = \tilde{A}\int_{-\infty}^{-4} e^{xt}\dfrac{(t+4)}{(t-3)}\,dt + \tilde{B}\oint_{C_2} e^{xt}(t+4)\dfrac{dt}{t-3}$

$t = -4-s$

$$= \tilde{A}\int_{\infty}^{0} e^{-4x} e^{-xs}\frac{(-s)(-dt)}{-s-7} + \frac{\tilde{B}}{2\pi i} e^{xt}(t+4)\Big|_{t=3} = \underline{Ae^{-4x}\int_0^{\infty} e^{-xt}\frac{t\,dt}{t+7} + Be^{3x}}$$

b) $y'' - xy = 1$ . Try $y = A(x)Y_1 + B(x)Y_2$ & so $y' = A'Y_1 + AY_1' + B'Y_2 + BY_2'$

$= AY_1' + BY_2'$ if $\left\{\begin{array}{l} A'Y_1 + B'Y_2 = 0 . \\ A'Y_1' + B'Y_2' = 1 \end{array}\right.$ Continuing $\cancel{AY_1''} + A'Y_1' + \cancel{BY_2''} + B'Y_2' - \cancel{xY_1A} - \cancel{xY_2B} = 1$

so

& $A'(Y_1Y_2' - Y_1'Y_2) = -Y_2$ , $B'(Y_2Y_1' - Y_2'Y_1) = -Y_1$

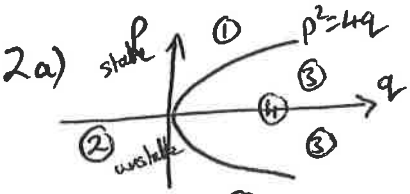
Now $\dfrac{d}{dx}(Y_2Y_1' - Y_2'Y_1) = \cancel{Y_2'Y_1'} + Y_2Y_1'' - Y_2''Y_1 - \cancel{Y_2'Y_1'} = xY_2Y_1 - xY_2Y_1 = 0$ so

$w = Y_2Y_1' - Y_2'Y_1 =$ Const, as evaluated at $x=0$ & $A' = Y_2/w$ , $B' = -Y_1/w$ .

$A = \dfrac{1}{w}\int_0^x Y_2\,ds + \tilde{A}$ , $B = \dfrac{1}{w}\int_0^x -Y_1\,ds + \tilde{B}$

$$y = AY_1 + BY_2 = \tilde{A}Y_1(x) + \tilde{B}Y_2(x) + \frac{1}{w}\int_0^x Y_1(x)Y_2(s) - Y_1(s)Y_2(x)\,ds$$

2a)




$\underline{x} = \begin{pmatrix}x\\y\end{pmatrix}$ , $\dot{\underline{x}} = \begin{pmatrix}A & B\\C & D\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$ . If $x = \underset{\lambda}{\check{x}} e^{\lambda t}$ , $\begin{vmatrix}A-\lambda & B\\C & D-\lambda\end{vmatrix} = 0$

$\lambda^2 - (A+D)\lambda + (AD-BC) = 0$

i) $\lambda$ real & same sign - a node ① - unstable if $\lambda > 0$ ie $p < 0$

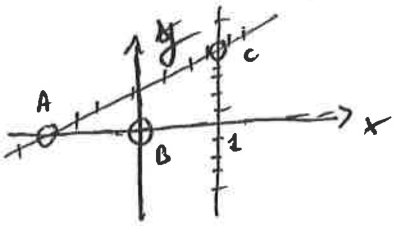
ii) $\lambda$ real & opposite sign - a saddle ②

iii) $\lambda$ complex - a spiral point ③ - unstable if $Re(\lambda) > 0$ ie $p < 0$ .

iv) $\lambda$ purely imaginary, a centre ④

$p = -(A+D)$
$q = AD - BC$

$\lambda$ are eigenvalues of $J = \begin{pmatrix}A & B\\C & D\end{pmatrix}$




$\dfrac{dx}{dt} = (1+x-2y)x = 0$ if $x=0$ or $y = \dfrac{1+x}{2}$ , vertical null clines

$\dfrac{dy}{dt} = (x-1)y = 0$ if $x=1$ or $y=0$ ; horizontal null clines

Critical points where these cross, ie $x=0, y=0$
$x=-1, y=0$
$x=1, y=1$

$\underline{\underline{J}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1+2x-2y & -2x \\ y & x-1 \end{pmatrix}$. At A, $x=-1, y=0$, $J = \begin{pmatrix} -1 & 2 \\ 0 & -2 \end{pmatrix}$ eigenvalues

are $-2$ & $-1$ $\Rightarrow$ stable node. Locally $dx/dt = -x+2y$, $\frac{dy}{dt} = -2y$, $\frac{dy}{dx} = \frac{-2y}{-x+2y}$ & if $y = mx$

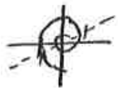
$m = \frac{-2m}{-1+2m}$, $2m^2 - m = -2m$, $m = 0$ or $-1/2$

At B, $x=0, y=0$, $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, eigenvalues $1$ & $-1$ $\Rightarrow$ saddle

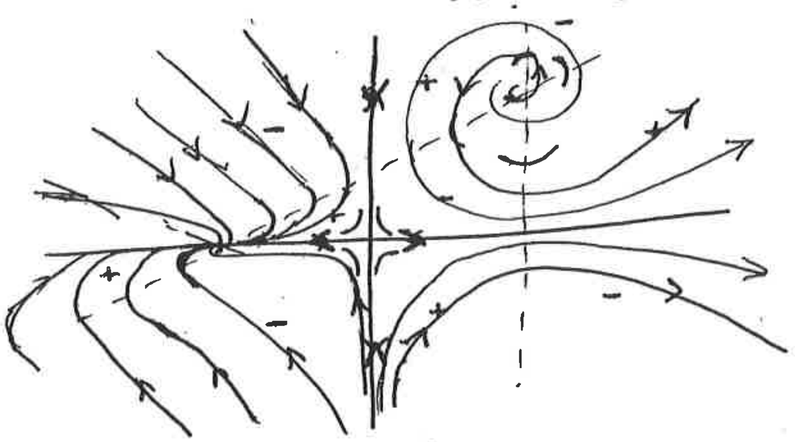
At C, $x=1, y=1$, $J = \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}$, eigenvalues have $(1-\lambda)(-\lambda) = -2$, $\lambda^2 - \lambda + 2 = 0$, $\lambda = \frac{1 \pm \sqrt{-7}}{2}$

$\Rightarrow$ unstable spiral

or



As $x \to +\infty$ for $y = O(1)$ $\frac{dy}{dt} \sim xy$, $\frac{dx}{dt} \sim x^2$, $\frac{dy}{dx} \sim \frac{y}{x} > 0$ $(y \sim ax)$



As $y \to \infty$ for $x = O(1)$

$\frac{dy}{dx} \sim (x-1)y$, $\frac{dy}{dx} \sim \frac{(x-1)}{-2x} \sim \frac{1}{2x}$ $x \to 0$

$\frac{dx}{dt} \sim -2yx$ $\sim -1/2$ $x \to \infty$

3) $\ddot{x} + \varepsilon f(x, \dot{x}) + x = 0$, $\frac{\partial}{\partial t} \to \frac{\partial}{\partial \theta} \cdot \frac{\partial \theta}{\partial t} = (1 + \varepsilon n_1)\frac{\partial}{\partial \theta}$, $\frac{\partial^2}{\partial t^2} \sim (1 + 2\varepsilon n_1)\frac{\partial^2}{\partial \theta^2}$

$\Rightarrow (1 + 2\varepsilon n_1 \ldots)(-a \sin\theta + \varepsilon x_1'') + \varepsilon f(a\sin\theta, (1+\ldots)a\cos\theta) + a\sin\theta + \varepsilon x_1 = 0$

$\Rightarrow x_1'' + x_1 = -f(a\sin\theta, a\cos\theta) + 2n_1 a\sin\theta$

Solutions remain periodic if coefficients of $\sin\theta$ & $\cos\theta$ in rhs are zero, ie

$-\int_0^{2\pi} f(a\sin\theta, a\cos\theta)\cos\theta\, d\theta + 2n_1 a \int_0^{2\pi} \underbrace{\sin\theta\cos\theta}_{0}\, d\theta = 0 \Rightarrow \int_{-\pi}^{\pi} f(a\sin\theta, a\cos\theta)\cos\theta\, d\theta = 0$

$-\int_0^{2\pi} f(a\sin\theta, a\cos\theta)\sin\theta\, d\theta + 2n_1 a \int_0^{2\pi} \sin^2\theta\, d\theta = 0 \Rightarrow n_1 = \frac{1}{2\pi a}\int_{-\pi}^{\pi} f(a\sin\theta, a\cos\theta)\sin\theta\, d\theta$

(Sean, bodworth)

b) Using above, with, $f(a\sin\theta, a\cos\theta) = (|a\sin\theta| - 1)|a\cos\theta| a\cos\theta + a^3\sin^3\theta$

gives $a^2\int_{-\pi}^{\pi} \cos^2\theta|\cos\theta|(a|\sin\theta| - 1) + \underbrace{a\sin^3\theta\cos\theta}_{\int \to 0}\, d\theta = 0 \Rightarrow a^{-1} = \frac{\int_{-\pi}^{\pi}\cos^2\theta|\cos\theta||\sin\theta|}{\int_{-\pi}^{\pi}\cos^2\theta|\cos\theta|\, d\theta}$

$= \frac{\int_0^{\pi/2}\cos^3\theta\sin\theta\, d\theta}{\int_0^{\pi/2}\cos^3\theta\, d\theta} = \frac{\left[-\frac{1}{4}\cos^4\theta\right]_0^{\pi/2}}{\int_0^{\pi/2}\cos\theta(1-\sin^2\theta)\, d\theta} = \frac{1/4}{1 - 1/3} = 3/8 \Rightarrow a = 8/3$

& $n_1 = \frac{1}{2\pi a}\int_{-\pi}^{\pi} \underbrace{(|a\sin\theta| - 1)|a\cos\theta|}_{\text{even}} a\underbrace{\cos\theta\sin\theta}_{\text{odd}} + a^3\sin^4\theta\, d\theta. = \frac{a^2}{2\pi}\cdot\frac{3\pi}{4} = \frac{8}{3}\cdot\frac{8}{3}\cdot\frac{3}{8} = 8/3$

So $x = 8/3\sin(1 + \varepsilon 8/3)t$, Period is $\frac{2\pi}{1 + 8\varepsilon/3} = 2\pi(1 - 8\varepsilon/3 \ldots)$

4) $x_{tt} + (a^2 + \varepsilon\cos 2t)x = 0$

a) $x_{1tt} + \alpha^2 x_1 = -A_0\cos(\alpha t)\cos 2t - B_0\sin(\alpha t)\cos 2t$

$$= -\frac{A_0}{2}\left\{\cos(\alpha-2)t + \cos(\alpha+2)t\right\} - \frac{B_0}{2}\left\{\sin(2+\alpha)t - \sin(2-\alpha)t\right\}$$

$x_1$ remains periodic unless $\alpha - 2 = \pm 1$ or $\alpha + 2 = \pm 1$ ie $\alpha = \pm 1$ or $\alpha = 1$, if $\alpha$ taken as positive.

b) $x_{tt} + x = -\varepsilon[2ax + \cos 2t\, x]$

$x = x_0(t,T) + \varepsilon x_1(t,T)$, $\frac{d}{dt} \to \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T}$, $\frac{d^2}{dt^2} \to \frac{\partial^2}{\partial t^2} + 2\frac{\partial}{\partial T}\frac{\partial}{\partial t} + \cdots$

$\Rightarrow x_{0tt} + x_0 = 0 \Rightarrow x_0 = A(T)\sin t + B(T)\cos t$

& $x_{1tt} + x_1 = -2\frac{\partial}{\partial T}\frac{\partial x_0}{\partial t} - 2ax_0 - \cos 2t\, x_0$

$$= -2\frac{\partial A}{\partial T}\cos t + 2\frac{\partial B}{\partial T}\sin t - 2a(A\sin t + B\cos t) - A\underbrace{(\sin t\cos 2t)}_{\frac12(\sin 3t - \sin t)} - B\underbrace{(\cos t\cos 2t)}_{\frac12(\cos t + \cos 3t)}$$

Set coefficients of $\sin t$ & $\cos t$ on rhs to zero

$(\cos t)$: $-2\frac{\partial A}{\partial T} - 2aB - \frac12 B = 0$ & $(\sin t)$ $2\frac{\partial B}{\partial T} - 2aA + \frac12 A = 0$

$$\Rightarrow \frac{dA}{dT} = -\left(\frac14 + a\right)B, \quad \frac{dB}{dT} = -\left(\frac14 - a\right)A$$

$\frac{d}{dT}\begin{pmatrix}A\\B\end{pmatrix} = \begin{pmatrix}0 & -(1/4+a)\\ -(1/4-a) & 0\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}$ & so, if $\begin{pmatrix}A\\B\end{pmatrix} = \underline{v}e^{\lambda t}$, $\det\begin{pmatrix}-\lambda & -(1/4+a)\\ -(1/4-a) & -\lambda\end{pmatrix} = 0$

ie $\lambda^2 = \frac{1}{16} - a^2$ & $\lambda = \pm\sqrt{\frac{1}{16} - a^2}$

If $|a| < 1/4$ then $\lambda$ real & amplitudes $A$ & $B$ grow on timescale $\varepsilon t \sim O(1)$

If $|a| > 1/4$ then $\lambda$ is purely imaginary & solution is periodic

5) a) If $f(x) \underset{x\to 0}{\sim} x^{\lambda_0}(a_0 + x^{\lambda_1}a_1 + \dots)$ then $\int_0^T e^{-xt}f(t)\,dt \sim \frac{a_0\lambda_0!}{x^{\lambda_0+1}} + \frac{a_1(\lambda_0+\lambda_1)!}{x^{\lambda_0+\lambda_1+1}} + \cdots$ as $x \to \infty$

& $f(x)$ does not grow superexponentially as $x \to \infty$

(Similar seen)

$\frac{1}{1+t^2} \underset{t\to 0}{\sim} 1 - t^2 + t^4$, so $\int_0^1 \frac{e^{-xt}}{1+t^2}dt \sim \frac{1}{x}\left(1 - \frac{2!}{x^2} + \frac{4!}{x^4}\right) = \frac{1}{x}\left(1 - \frac{2}{x^2} + \frac{24}{x^4}\right)$

b) Use the method of Stationary Phase, $\int_a^b e^{ixf(t)}g(t)\,dt \sim e^{ixf(t_0)}g(t_0)\sqrt{\frac{2}{x|f''(t_0)|}}\,e^{i\,\mathrm{sgn} f''(t_0)}$

where $f'(t_0) = 0$. Here, writing $t = x^{1/2}u$, we have

$I = \frac{\sqrt{x}}{\pi}\int_0^\infty \sin x^{3/2}\left(\frac{u^3}{3} - u\right)du = \frac{\sqrt{x}}{\pi}\,\mathrm{Im}\int_0^\infty e^{ix^{3/2}(u^3/3 - u)}du$. $f(u) = u^3/3 - u$, $f'(u) = u^2 - 1$

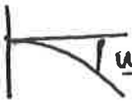
$\Rightarrow u_0 = 1$, $f'' = 2u$, so $f''(u_0) = 2$, $f(u_0) = -2/3$ & $I \approx \frac{\sqrt{x}}{\pi}\,\mathrm{Im}\; e^{-ix^{3/2}\cdot 2/3}\cdot 1\sqrt{\frac{2}{x^{3/2}\cdot 2}}\,e^{i\pi/4}$

$$= \frac{1}{\sqrt{\pi}\,x^{1/4}}\sin\left(\pi/4 - \frac{2x^{3/2}}{3}\right)$$

c)

$$\int_0^\infty e^{-xt}(1+t)^x\,dt = \int_0^\infty e^{-xt}e^{x\ln(1+t)}\,dt = \int_0^\infty e^{x[\ln(1+t)-t]}\,dt$$

$\varphi(t) = \ln(1+t) - t$



$u = t - \ln(1+t)$

$du = 1 - \frac{1}{1+t}\,dt$

$= \frac{t}{1+t}\,dt$

$$= \int_0^\infty e^{-xu}\left(1+\frac{1}{t}\right)du, \quad t = t(u)$$

For small $t$ $u \sim t^2/2$, so $t \approx \sqrt{2u}$, but this is not sufficient. If $t \sim \sqrt{2u} + \alpha u + \dots$ then

$u \sim \sqrt{2u} + \alpha u + \dots - \ln(1 + \sqrt{2u} + \alpha u \dots)$

$= \cancel{\sqrt{2u}} + \cancel{\alpha u} + \dots - (\cancel{\sqrt{2u}} + \cancel{\alpha u} \dots) + \frac{1}{2}\left(2u + 2\sqrt{2}\alpha u^{3/2} + \dots\right)$

$-\frac{1}{3}\left(2\sqrt{2}\,u^{3/2}\dots\right)$

$\Rightarrow \alpha = 2/3$

& $1 + \frac{1}{t} \sim 1 + \frac{1}{\sqrt{2u} + \frac{2u}{3}} \dots = 1 + \frac{1}{\sqrt{2u}}\frac{1}{1 + \frac{\sqrt{2u}}{3}} \sim 1 + \frac{1}{\sqrt{2u}}\left(1 - \frac{\sqrt{2u}}{3}\right) \sim \frac{1}{\sqrt{2u}} + \frac{2}{3}$

& $I \sim \sqrt{\frac{\pi}{2x}} + \frac{2}{3x}$